Mean square exponential stability of numerical methods for stochastic differential delay equations

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1. Backgrounds

Consider the nonlinear SDDE

$$dx(t) = f(x(t), x(t - \tau(t)))dt + g(x(t), x(t - \tau(t)))dB(t) \quad (1)$$

with initial value ξ, $B(t) := (B_1(t), \ldots, B_m(t))^T$ be an m-dimensional standard Brownian motion, $f : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d, \quad g : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^{d \times m}$ and $\tau(t) : \mathbb{R}_+ \to \mathbb{R}_+$ are Borel-measurable functions and $0 < \tau(t) \leq \tau, t \in \mathbb{R}_+$, for some $\tau > 0$.

Asymptotic stability of numerical approximations for the above SDDE (1) or more general model has been widely investigated in recent years. In general, to obtain the exponential stability results, the following Khasminskii-type conditions are usually parts of the sufficient conditions.

$$2\langle x, f(x,y)\rangle + ||g(x,y)||^2 \leq -C_1|x|^2 + C_2|y|^2 \tag{2}$$

where $C_1 > C_2 > 0$.

For example, in [X. Mao, Horwood, Chichester, 2007], Chapter 5, Razumikhin type theorems are presented for SDDEs, where conditions in Corollary 6.6 implies (2), in [G. Lan, J. Comput. Appl. Math., 340(2018), 334-341], Theorem 4.1 implied that if there is no neutral term, then the MTEM method is mean square exponentially stable under condition (2), under the same condition, [Y. Zhang, M. Song, M, Liu, J. Comput. Appl. Math., 403(2022), 113849] obtained exponential stability of stochastic theta method for nonlinear stochastic differential equations with piecewise continuous arguments (in this case, $\tau(t) = t - [t]$).

[L. Liu, H. Mo, F. Deng, Appl. Math. Comput., 353(2019) 320-328] obtained mean square exponential stability of split-step method under stronger conditions (note that they need the linearity of both f and g), while [L. Liu, Q. Zhu, J. Comput. Appl. Math., 305(2016) 55-67] and [M. Obradović, M. Milošević, Calcolo, 56:2(2019) 1-24] investigated mean square stability of two class of theta method of neutral stochastic differential delay equations under similar conditions.

Recently, by considering each component separately, [P. H. A. Ngoc, L. T. Hieu, IEEE Trans. Automat. Control, 66(2021), 2351-2356] presented a different type of sufficient conditions under which the trivial solution of given SDDE is mean square exponentially stable. However, they need the diffusion term g satisfy linear growth condition.

Motivated by this paper, now suppose that f and g jointly satisfy the following

$$2x_i f_i(x,y) + \sum_{l=1}^{m} (g_{il}(x,y))^2 \le \sum_{j=1}^{d} a_{ij} x_j^2 + \sum_{j=1}^{d} b_{ij} y_j^2, \quad i \in \underline{d} \quad (3)$$

where $\underline{d} := \{1, 2, ..., d\}$.

Note that by (3) we can only have

$$2\langle x, f(x,y)\rangle + ||g(x,y)||^2 \le \sum_{j=1}^{d} \sum_{i=1}^{d} a_{ij} x_j^2 + \sum_{j=1}^{d} \sum_{i=1}^{d} b_{ij} y_j^2.$$

If $\sum_{i=1}^{d} a_{ij} \ge 0$ for some $j \in \underline{d}$, or $0 < \min_{j\in\underline{d}}(-\sum_{i=1}^{d} a_{ij}) < \max_{j\in\underline{d}} \sum_{i=1}^{d} b_{ij}$, then (2) can never hold. Therefore, (3) is a totally different type condition from (2).

To the best of our knowledge, *no results can be found for the exponential stability of numerical methods with condition (3).*

In this talk, we will first define two numerical methods (including θ-EM method and MTEM method) for SDDEs, and then we will investigate the asymptotic mean square exponential stability of the two given methods.

2. Settings and θ-EM method and MTEM method

Let $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, P)$ be a complete filtered probability space. Let us give the definition of the mean square exponential stability of SDDE (1).

Definition 1: The solution $x(t)$ of SDDE (1) is said to be mean square exponentially stable if there exists a pair of positive constants λ and C such that

$$E|x(t)|^2 \leq CE|x_0|^2 e^{-\lambda t} \tag{4}$$

for all initial value $x_0 \in \mathbb{R}^d$.

To investigate mean square exponential stability of solution $x(t)$ and the corresponding numerical methods of SDDE (1), suppose $f(0,0) = 0, g(0,0) = 0$. We also need the following assumptions:

Assumption 1: Assume that both the coefficients f and g in (1) are locally Lipschitz continuous, that is, for each $R > 0$ there is $L_R > 0$ (depending on R) such that

$$|f(x,y) - f(\bar{x},\bar{y})| \vee |g(x,y) - g(\bar{x},\bar{y})| \leq L_R(|x-\bar{x}| + |y-\bar{y}|) \quad (5)$$

for all $|x| \vee |y| \vee |\bar{x}| \vee |\bar{y}| \leq R > 0$.

Let Δ be a stepsize such that $\tau = \bar{m}\Delta$ for some positive integer $\bar{m}$. Then for any $\theta \in [0,1]$, we can define θ Euler-Maruyama method (θ-EM for short) X_k as the following:

$$\begin{aligned} X_k &= \xi(k\Delta), \quad k = -\bar{m}, -\bar{m}+1, \dots, 0. \\ X_{k+1} &= X_k + \left((1-\theta) f\left(X_k, X_{k-[\frac{\tau(k\Delta)}{\Delta}]}\right)\right. \\ &\quad \left. + \theta f\left(X_{k+1}, X_{k+1-[\frac{\tau((k+1)\Delta)}{\Delta}]}\right)\right)\Delta \\ &\quad + g\left(X_k, X_{k-[\frac{\tau(k\Delta)}{\Delta}]}\right)\Delta B_k, \quad k = 0, 1, 2, \dots, \end{aligned} \tag{6}$$

Here $\Delta B_k = B((k+1)\Delta) - B(k\Delta)$ is the increment of the m-dimensional standard Brownian motion.

Note that if $\theta = 0$, it becomes to classical EM method, if $\theta = 1$, it is called Backward Euler method. Moreover, since it is an implicit method for $\theta \in (0, 1]$, then it is necessary to make sure that θ-EM method is well defined. So we need the following assumption:

Assumption 2 Assume that f is one-sided Lipschitz continuous, that is, there is $L > 0$ such that

$$\langle x - \bar{x}, f(x, y) - f(\bar{x}, y) \rangle \leq L|x - \bar{x}|^2 \tag{7}$$

for all $x, y, \bar{x} \in \mathbb{R}^d$.

According to [F. Wu, X. Mao, L. Szpruch, Numer. Math. 115 (2010)681-697], [E. Hairer, G. Wanner, Springer, Berlin, 1996] or [X. Mao, C. Yuan, Imperial College Press, London, 2006], if $L\theta\Delta < 1$, then the θ-EM scheme (6) is well defined.

Now let us consider the so called MTEM method, which is an explicit method.

For $\Delta^* > 0$, let $h(\Delta)$ be a strictly positive decreasing function $h : (0, \Delta^*] \to (0, \infty)$ such that

$$\lim_{\Delta \to 0} h(\Delta) = \infty \text{ and } \lim_{\Delta \to 0} L^2_{h(\Delta)} \Delta = 0.$$

According to [G. Lan, F. Xia, J. Comput. Appl. Math., 334(2018), 1-17.], Remark 2.1, such h always exists.

For any $\Delta \in (0, \Delta^*)$, we define the modified truncated function of f as the following:

$$f_\Delta(x,y) = \begin{cases} f(x,y), & |x| \vee |y| \le h(\Delta), \\ \frac{|x|\vee|y|}{h(\Delta)} f\left(\frac{h(\Delta)}{|x|\vee|y|}(x,y)\right), & |x| \vee |y| > h(\Delta). \end{cases}$$

g_Δ is defined in the same way as f_Δ. Here $f(a(x,y)) \equiv f(ax, ay)$ for any $t \ge 0, a \in (0,1), x, y \in \mathbb{R}^d$.

Let Δ be a stepsize such that $\tau = \bar{m}\Delta$ for some positive integer $\bar{m}$. Then by using f_Δ and g_Δ, we can define the modified truncated Euler-Maruyama method (MTEM for short) X_k as the following:

$$\begin{aligned} X_{k+1} = X_k + f_\Delta\left(X_k, X_{k-[\frac{\tau(k\Delta)}{\Delta}]}\right)\Delta \\ + g_\Delta\left(X_k, X_{k-[\frac{\tau(k\Delta)}{\Delta}]}\right)\Delta B_k, k \geq 0, \\ X_k = \xi(k\Delta), \quad k = -\bar{m}, -\bar{m}+1, \dots, 0. \end{aligned} \tag{8}$$

Here $\Delta B_k = B((k+1)\Delta) - B(k\Delta)$ is the increment of the m-dimensional standard Brownian motion.

Assumption 3:
Let $f(x,y) := (f_1(x,y), f_2(x,y), \ldots, f_d(x,y))^T \in \mathbb{R}^d$ and $g(x,y) := (g_{il}(x,y)) \in \mathbb{R}^{d\times m}$. Suppose there exist constants $a_{ii} \in \mathbb{R}$; $a_{ij} \geq 0, i \neq j$; $b_{ij} \geqslant 0,\ i,j \in \underline{d}$, such that

$$2x_i f_i(x,y) + \sum_{l=1}^{m} (g_{il}(x,y))^2 \leq \sum_{j=1}^{d} a_{ij} x_j^2 + \sum_{j=1}^{d} b_{ij} y_j^2 \qquad (9)$$

holds for any $i \in \underline{d}$. And there exist constants $p_j > 0, j \in \underline{d}$ such that

$$\sum_{j=1}^{d} (a_{ij} + b_{ij}) p_j < 0. \qquad (10)$$

Definition 2: The numerical method X_k is said to be mean square (or almost surely) exponential stable with rate $\lambda > 0$ if there exists $\Delta^* > 0$ such that for any $\Delta \in (0, \Delta^*)$ the discrete approximation satisfies

$$\limsup_{k\to\infty} \frac{\log \mathbb{E}|X_k|^2}{k\Delta} \leq -\lambda \quad (\text{or} \quad \limsup_{k\to\infty} \frac{\log |X_k|}{k\Delta} \leq -\lambda \quad a.s.). \tag{11}$$

Firstly, we have the following exponential stability result for SDDE (1).

Theorem 1 Suppose Assumptions 1 and 3 hold. Then for any given initial data ξ, there always exists a unique solution $x(t, \xi)$ and the trivial solution of (1) is exponentially stable in mean square sense.

Remark 1 Notice that existence and uniqueness of solution to equation (1) is obvious since local Lipschitz continuity of f and g implies there exists a unique (local) solution to equation (1), and Assumption 3 implies

$$2\langle x, f(x,y)\rangle + ||g(x,y)||^2 \leq K(1+|x|^2+|y|^2),$$

which guarantees the unique solution is global. Moreover, it is obvious that Assumption 2 includes the case that the noise term g is not linear growing with respect to space value. On the other hand, in [P. H. A. Ngoc, L. T. Hieu, IEEE Trans. Automat. Control, 66(2021), 2351-2356] the diffusion term g must be linear growing. So our results is a generalization of Theorem II.2 in Ngoc and Hieu.

3. Exponential stability of θ-EM method

Theorem 2 Suppose Assumptions 1-3 hold. Moreover, there exist constants $0 < \varepsilon < \frac{\min_i |a_{ii}|}{d \max_i |a_{ii}|}$ and $p_j > 0, j \in \underline{d}$ such that

$$\varepsilon a_{ii} p_i + \sum_{j \neq i} a_{ij} p_j + \sum_{j=1}^{d} b_{ij} p_j < 0. \tag{12}$$

Then for any fixed $\theta \in (\frac{1}{2}, 1]$, the θ-EM method (6) is mean quare and almost surely exponentially stable.

Remark 2 Notice that (12) implies

$$\sum_{j=1}^{d} a_{ij} p_j + \sum_{j=1}^{d} b_{ij} p_j < 0. \tag{13}$$

Then similar to the proof of Theorem II.2 in [Ngoc, Hieu, IEEE Trans. Automat. Control, 66(2021), 2351-2356], it follows that the Assumption 3 implies that the solution $x(t, \xi)$ to (1) is exponentially stable in mean square sense if (10) holds. Theorem 2 assures that θ-EM method (6) replicates mean square exponential stability of the exact solution under given conditions.

Theorem 3 Suppose all conditions in Theorem 2 hold, and there exists a constant $K > 0$ such that

$$|f(x, y)| \leq K(|x| + |y|). \tag{14}$$

Then for any fixed $0 \leq \theta \leq \frac{1}{2}$, θ-EM method is mean square and almost surely exponentially stable for sufficient small step size $\Delta > 0$.

Sketch of the Proof of Theorem 2 and 3.

Define $F_k^i := X_k^i - \theta f_i \left(X_k, X_{k-[\frac{\tau(k\Delta)}{\Delta}]} \right) \Delta$,
$f_k^i := f_i \left(X_k, X_{k-[\frac{\tau(k\Delta)}{\Delta}]} \right), g_k^{il} := g_{il} \left(X_k, X_{k-[\frac{\tau(k\Delta)}{\Delta}]} \right)$.
The main idea is second principle of mathematical induction.
It is obvious that

$$E|X_k^i|^2 \leq C' p_i e^{-\beta k \Delta} \tag{15}$$

holds for $-m \leq k \leq 0, i = 1, 2, ..., d$.

Assume that

$$E|X_l^i|^2 \leq C' p_i e^{-\beta l \Delta}, \forall -\bar{m} \leq l \leq k-1, i = 1, 2, ..., d. \qquad (16)$$

By using

$$|F_{k+1}^i|^2 \leq (1-C\Delta)|F_k^i|^2 + (\sum_{j \neq i}^{d} a_{ij}|X_k^j|^2 + \sum_{j=1}^{d} b_{ij}|X_{k-[\frac{\tau(k\Delta)}{\Delta}]}^j|^2)\Delta + M_k^i$$

and the fact that there exist $\beta > 0$ small enough and $\varepsilon' > 0$ such that $\varepsilon \frac{\max_i |a_{ii}|}{\min_i |a_{ii}|} < \varepsilon' < \frac{1}{d}$ and

$$\varepsilon a_{ii} p_i + \sum_{j \neq i} a_{ij} p_j + \sum_{j=1}^{d} b_{ij} p_j e^{\beta \tau} \leq -\varepsilon' \beta p_i \qquad (17)$$

for any $i \in \underline{d}$.
We firstly derive that

$$E|F_k^i|^2 \leq C' \varepsilon' p_i e^{-\beta k \Delta}. \qquad (18)$$

Then by

$$
\begin{aligned}
|F_k^i|^2 &= |X_k^i|^2 - 2\theta\Delta X_k^i f_k^i + \theta^2\Delta^2 |f_k^i|^2 \\
&\geq |X_k^i|^2 - \theta\Delta\left(\sum_{j=1}^d a_{ij}|X_k^j|^2 + \sum_{j=1}^d b_{ij}\left|X^j_{k-[\frac{\tau(k\Delta)}{\Delta}]}\right|^2\right),
\end{aligned}
$$

we can finally derive

$$
\begin{aligned}
\sum_{i=1}^d \frac{|X_k^i|^2}{p_i} \leq \frac{1}{1-D}\Bigg(&\sum_{i=1}^d \frac{|F_k^i|^2}{p_i(1-\theta\Delta a_{ii})} \\
&+ \sum_{j=1}^d \left(\sum_{i=1}^d \frac{\theta\Delta}{1-\theta\Delta a_{ii}} \frac{b_{ij}p_j}{p_i}\right) \frac{\left|X^j_{k-[\frac{\tau(k\Delta)}{\Delta}]}\right|^2}{p_j}\Bigg)
\end{aligned}
$$

where $D = \sum_{i=1}^d \frac{\theta\Delta}{1-\theta\Delta a_{ii}} \max_{j\neq i} \frac{a_{ij}p_j}{p_i} < 1$ for sufficiently small Δ.

It follows that

$$\sum_{i=1}^{d} \frac{E|X_k^i|^2}{p_i} \le \frac{1}{1-D} \left(\frac{dC'\varepsilon' e^{-\beta k\Delta}}{1-\theta\Delta \max_i |a_{ii}|} \right.$$

$$\left. + C' e^{-\beta k\Delta} e^{\beta\tau} \sum_{j=1}^{d} \sum_{i=1}^{d} \frac{\theta\Delta}{1-\theta\Delta a_{ii}} \frac{b_{ij} p_j}{p_i} \right)$$

$$= C' e^{-\beta k\Delta} \frac{1}{1-D} \left(\frac{d\varepsilon'}{1-\theta\Delta \max_i |a_{ii}|} \right.$$

$$\left. + e^{\beta\tau} \sum_{j=1}^{d} \sum_{i=1}^{d} \frac{\theta\Delta}{1-\theta\Delta a_{ii}} \frac{b_{ij} p_j}{p_i} \right).$$

$$=: C' e^{-\beta k\Delta} G(\Delta).$$

Since $G(0) = d\varepsilon' < 1$, and G is continuous, then there exists Δ^* small enough such that $G(\Delta) \le 1$ for all $\Delta \le \Delta^*$.

If f satisfies linear growth condition, then

$$|X_k^i|^2 \geq \frac{|F_k^i|^2 - (\theta\Delta K + 2\theta^2\Delta^2 K^2)(\sum_{j\neq i}^d |X_k^j|^2 + |X_{k-[\frac{\tau(k\Delta)}{\Delta}]}|^2)}{1 + 4\theta\Delta K + 2\theta^2\Delta^2 K^2}.$$

Thus

$$\begin{aligned}|F_{k+1}^i|^2 \leq |F_k^i|^2 &+ (2(1-2\theta)K^2\Delta + a_{ii})\Delta \\ &\times \frac{|F_k^i|^2 - (\theta\Delta K + 2\theta^2\Delta^2 K^2)(\sum_{j\neq i}^d |X_k^j|^2 + |X_{k-[\frac{\tau(k\Delta)}{\Delta}]}|^2)}{1 + 4\theta\Delta K + 2\theta^2\Delta^2 K^2} \\ &+ \sum_{j\neq i}(2(1-2\theta)K^2\Delta + a_{ij})|X_k^j|^2\Delta \\ &+ \sum_{j=1}^d (2(1-2\theta)K^2\Delta + b_{ij})|X^j_{k-[\frac{\tau(k\Delta)}{\Delta}]}|^2\Delta + M_k^i.\end{aligned}$$

Or

$$|F^i_{k+1}|^2 \leq |F^i_k|^2\,(1 + C_{1,\Delta}\Delta) + \sum_{j\neq i}(C_{2,\Delta} + a_{ij})|X^i_k|^2\Delta$$

$$+ \sum_{j=1}^{d}(C_{2,\Delta} + b_{ij})\,|X^i_{k-[\frac{\tau(k\Delta)}{\Delta}]}|^2\Delta + M^i_k$$

where it is obvious that

$$C_{1,\Delta} := \frac{2(1-2\theta)K^2\Delta + a_{ii}}{1 + 4\theta\Delta K + 2\theta^2\Delta^2K^2} \to a_{ii}$$

and

$$C_{2,\Delta} := 2(1-2\theta)K^2\Delta - (\theta K + 2\theta^2\Delta K^2)\frac{(2(1-2\theta)K^2\Delta + a_{ii})\Delta}{1 + 4\theta\Delta K + 2\theta^2\Delta^2K^2} \to 0$$

as $\Delta \to 0$.

Then for any $C < \min_i |a_{ii}|$, we can choose $\tilde{\varepsilon} > 0$ and $\Delta > 0$ small enough such that

$$|F^i_{k+1}|^2 \leq |F^i_k|^2(1-C\Delta)+\sum_{j\neq i}(a_{ij}+\tilde{\varepsilon})|X^j_k|^2\Delta+\sum_{j=1}^{d}(b_{ij}+\tilde{\varepsilon})|X^j_{k-[\frac{\tau(k\Delta)}{\Delta}]}|^2+M^i_k$$

Note that (12) implies that there exist $\beta > 0$ small enough and $\varepsilon' > 0$ such that $\varepsilon\frac{\max_i |a_{ii}|}{\min_i |a_{ii}|} < \varepsilon' < \frac{1}{d}$ and

$$\varepsilon a_{ii}p_i + \sum_{j\neq i}(a_{ij}+\tilde{\varepsilon})p_j + \sum_{j=1}^{d}(b_{ij}+\tilde{\varepsilon})p_j e^{\beta\tau} \leq -\varepsilon'\beta p_i \qquad (19)$$

for any $i \in \underline{d}$.

That is, (17) holds for a_{ij} and b_{ij} replaced by $a_{ij}+\tilde{\varepsilon}$ and $b_{ij}+\tilde{\varepsilon}$, respectively.

Then repeat the following part of the proof of Theorem 2 from line to line, we complete the proof of Theorem 3.

4. Exponential stability of MTEM method

To obtain the mean exponential stability of MTEM method, we need the following two Lemmas.
Lemma 1: Suppose the Assumption 1 and Assumption 2 hold. Then for any fixed $\Delta > 0$,there exist constant matrices $a_{ii} \in \mathbb{R}; a_{ij} \geq 0, i \neq j; b_{ij} \geq 0.\ i \in \underline{d}$, such that

$$2x_i \cdot f_{\Delta,i}(x,y) + \sum_{l=1}^{m} g_{\Delta,il}^2(x,y) \leq \sum_{j=1}^{d} a_{ij} x_j^2 + \sum_{j=1}^{d} b_{ij} y_j^2, \quad i \in \underline{d} \quad (20)$$

for any x, y.

Lemma 2: Suppose Assumption 1 holds. Then for any fixed $\Delta > 0$, the modified truncated functions f_Δ and g_Δ are linear growing with coefficient $L_{h(\Delta)}$. That is

$$|f_\Delta(x, y)| \vee |g_\Delta(x, y)| \leq L_{h(\Delta)}(|x| + |y|) \tag{21}$$

holds for all $x, y \in \mathbb{R}^d$.

Remark In [G. Lan, J. Comput. Appl. Math., 340(2018), 334-341], the author proved that both f_Δ and g_Δ are globally Lipschitz continuous for any fixed $\Delta > 0$. However, we can only obtain $|f_\Delta(x, y)| \leq 5L_{h(\Delta)}(|x| + |y|)$ while by Lemma 2, we have $|f_\Delta(x, y)| \leq L_{h(\Delta)}(|x| + |y|)$.

Now we are ready to present the main result in this section.

Theorem 4 Suppose Assumptions 1 and 3 hold. Then the MTEM method (8) is mean quare and almost surely exponentially stable.

Sketch of the Proof of Theorem 4.

By (8), for any $k \geqslant 0$, we have

$$
\begin{aligned}
|X_{k+1}^i|^2 = |X_k^i|^2 + \Bigg(2X_k^i f_{\Delta,i}\left(X_k, X_{k-[\frac{\tau(k\Delta)}{\Delta}]}\right) \\
+ \sum_{l=1}^{m} g_{\Delta,il}^2\left(X_k, X_{k-[\frac{\tau(k\Delta)}{\Delta}]}\right)\Bigg)\Delta \\
+ f_{\Delta,i}^2\left(X_k, X_{k-[\frac{\tau(k\Delta)}{\Delta}]}\right)\Delta^2 + M_k
\end{aligned}
\tag{22}
$$

where

$$
\begin{aligned}
M_k = {} & 2\left\langle X_k^i, \sum_{l=1}^{m} g_{\Delta,il}\left(X_k, X_{k-[\frac{\tau(k\Delta)}{\Delta}]}\right)\Delta B_k \right\rangle \\
& + 2\left\langle f_{\Delta,i}\left(X_k, X_{k-[\frac{\tau(k\Delta)}{\Delta}]}\right), \sum_{l=1}^{m} g_{\Delta,il}\left(X_k, X_{k-[\frac{\tau(k\Delta)}{\Delta}]}\right)\Delta B_k \right\rangle \Delta \\
& + \left(\left| \sum_{l=1}^{m} g_{\Delta,il}\left(X_k, X_{k-[\frac{\tau(k\Delta)}{\Delta}]}\right)\Delta B_k \right|^2 \right. \\
& \left. - \left| \sum_{l=1}^{m} g_{\Delta,il}\left(X_k, X_{k-[\frac{\tau(k\Delta)}{\Delta}]}\right)\right|^2 \Delta \right).
\end{aligned}
$$

It is obvious that M_k is a $\mathscr{F}_{k\Delta}$ martingale and $EM_k = 0$. Then, by Lemmas 1 and 2, we have

$$\begin{aligned}
& e^{\alpha(k+1)\Delta} E|X_{k+1}^i|^2 - e^{\alpha k\Delta} E|X_k^i|^2 \\
\leq\ & e^{\alpha(k+1)\Delta}(1 - e^{-\alpha\Delta}) E|X_k^i|^2 \\
& + \sum_{j=1}^{d} (a_{ij} + 2L_{h(\Delta)}^2 \Delta) E|X_k^j|^2 e^{\alpha(k+1)\Delta} \Delta \\
& + (\sum_{j=1}^{d} b_{ij} + 2L_{h(\Delta)}^2 \Delta) E|X_{k-[\frac{\tau(k\Delta)}{\Delta}]}^j|^2 e^{\alpha(k+1)\Delta} \Delta.
\end{aligned}$$

Since (10) holds, similar to the proof of Theorem 1, there exist $\beta > 0$ and $\varepsilon > 0$ small enough such that

$$\sum_{j=1}^{d} \left(a_{ij} + \varepsilon + e^{\beta\tau} (b_{ij} + \varepsilon) \right) p_j \leq -\beta p_i \tag{23}$$

for any $i \in \underline{d}$

Assume that

$$E|X_k^i|^2 \le \bar{K} p_i e^{-\beta k \Delta} E||\xi||^2, \quad k \leqslant n-1. \tag{24}$$

Now for $k = n$, it follows that

$$\begin{aligned} e^{\alpha n \Delta} E|X_n^i|^2 \le E|X_0^i|^2 &+ \sum_{l=0}^{n-1} e^{\alpha(l+1)\Delta}(1 - e^{-\alpha\Delta} \\ &+ (a_{ii} + 2L_{h(\Delta)}^2 \Delta)\Delta) E|X_l^i|^2 \\ &+ \sum_{l=0}^{n-1} \sum_{i \neq j} (a_{ij} + 2L_{h(\Delta)}^2 \Delta)\Delta e^{\alpha(l+1)\Delta} E|X_l^j|^2 \\ &+ \sum_{l=0}^{n-1} \sum_{j=1}^{d} (b_{ij} + 2L_{h(\Delta)}^2 \Delta)\Delta e^{\alpha(l+1)\Delta} E|X_{l-[\frac{\tau(l\Delta)}{\Delta}]}^j|^2. \end{aligned}$$

Choose $\alpha > \max_{i \in \underline{d}} |a_{ii}|$, then $1 - e^{-\alpha\Delta} + (a_{ii} + 2L^2_{h(\Delta)}\Delta)\Delta > 0$.

$$
\begin{aligned}
& e^{\alpha n\Delta} E|X^i_n|^2 \\
\leq\ & E|X^i_0|^2 + \sum_{l=0}^{n-1} e^{\alpha(l+1)\Delta}(1 - e^{-\alpha\Delta} + (a_{ii} + 2L^2_{h(\Delta)}\Delta)\Delta) K_1 p_i e^{-\beta l\Delta} \\
& + \sum_{l=0}^{n-1} \sum_{i \neq j} (a_{ij} + 2L^2_{h(\Delta)}\Delta)\Delta e^{\alpha(l+1)\Delta} K_1 p_j e^{-\beta l\Delta} \\
& + \sum_{l=0}^{n-1} \sum_{j=1}^{d} (b_{ij} + 2L^2_{h(\Delta)}\Delta)\Delta e^{\alpha(l+1)\Delta} K_1 p_j e^{-\beta(l - [\frac{\tau(l\Delta)}{\Delta}])\Delta} \\
\leq\ & E|X^i_0|^2 + \sum_{l=0}^{n-1} (e^{\alpha\Delta} - 1) K_1 p_i \cdot e^{(\alpha-\beta)l\Delta} \\
& + \sum_{l=0}^{n-1} \sum_{j=1}^{d} ((a_{ij} + 2L^2_{h(\Delta)}\Delta) + (b_{ij} + 2L^2_{h(\Delta)}\Delta) e^{\beta\tau}) p_j \Delta K_1 e^{\alpha\Delta} e^{(\alpha-\beta)l\Delta}.
\end{aligned}
$$

where $K_1 = \bar{K} E||\xi||^2$.

Then if we choose the stepsize $\Delta > 0$ sufficiently small such that $2L^2_{h(\Delta)}\Delta \le \varepsilon$, (23) yields

$$
\begin{aligned}
& e^{\alpha n\Delta} E|X^i_n|^2 \\
\le\ & E|X^i_0|^2 + K_1 p_i \cdot (e^{\alpha\Delta} - 1) \cdot \frac{e^{(\alpha-\beta)n\Delta} - 1}{e^{(\alpha-\beta)\Delta} - 1} \\
& - K_1 \beta p_i \Delta e^{\alpha\Delta} \frac{e^{(\alpha-\beta)n\Delta} - 1}{e^{(\alpha-\beta)\Delta} - 1} \\
=\ & E|X^i_0|^2 + K_1 p_i \cdot (e^{\alpha\Delta} - 1 - \beta\Delta e^{\alpha\Delta}) \cdot \frac{e^{(\alpha-\beta)n\Delta} - 1}{e^{(\alpha-\beta)\Delta} - 1}.
\end{aligned}
$$

On the other hand, it is obvious that

$$\begin{aligned} e^{\alpha\Delta} - 1 - \beta\Delta e^{\alpha\Delta} &= (1-\beta\Delta)e^{\alpha\Delta} - 1 \\ &\le e^{-\beta\Delta}\cdot e^{\alpha\Delta} - 1 = e^{(\alpha-\beta)\Delta} - 1. \end{aligned} \tag{25}$$

Therefore,

$$\begin{aligned} e^{\alpha n\Delta}E|X_n^i|^2 &\le E|X_0^i|^2 + K_1 p_i\cdot(e^{(\alpha-\beta)n\Delta} - 1) \\ &= E|X_0^i|^2 - K_1 p_i + K_1 p_i\cdot e^{(\alpha-\beta)n\Delta} \\ &\le K_1 p_i\cdot e^{(\alpha-\beta)n\Delta}. \end{aligned} \tag{26}$$

Thus,

$$E|X_n^i|^2 \leq K_1 p_i \cdot e^{-\beta n\Delta}. \tag{27}$$

For almost sure exponential stability, a standard procedure of using Chebyshev inequality and Borel-Cantelli's Lemma implies that

$$\limsup_{n\to\infty} \frac{\log|X_n^i|}{n\Delta} \leq -\frac{\beta}{2}, a.s.$$

5. Examples

Example 1 Consider a 2-D stochastic differential delay equation given by

$$
\begin{aligned}
dx_1(t) &= (-\frac{1}{5}x_1(t) - \frac{2}{9}x_1^3(t) + \frac{4}{5}x_2(t) + \frac{1}{10^4}x_1(t-\tau(t)) \\
&\quad + \frac{1}{10^4}x_2(t-\tau(t)))dt + \frac{2}{3}x_1^2(t)dB_1(t) \\
dx_2(t) &= (\frac{\sqrt{38}}{625}x_1(t) - x_2(t) + \frac{1}{10^4}x_1(t-\tau(t)) \\
&\quad + \frac{1}{10^4}x_2(t-\tau(t)))dt + \sqrt{\frac{2}{5}}x_2(t)dB_2(t)
\end{aligned}
\tag{28}
$$

for $t \geq 0$, where the initial value $\xi(s) \in C([-\tau, 0], \mathbb{R}^d)$, $\tau(t) = 0.1(1 - |\sin(t)|) \leq \tau = 0.1$, and $B(\cdot) = (B_1(\cdot), B_2(\cdot))^T$ is a 2-D Brownian motion.

Clearly

$$\begin{aligned}
&2x_1 f_1(x,y) + g_{11}^2(x,y) + g_{12}^2(x,y) \\
&= -\frac{2}{5}x_1^2 - \frac{4}{9}x_1 x_1^3 + \frac{8}{5}x_1 x_2 + \frac{2}{10^4}x_1 y_1 + \frac{2}{10^4}x_1 y_2 + \frac{4}{9}x_1^4 \\
&\le -\frac{399}{5000}x_1^2 + 2x_2^2 + \frac{1}{10^4}y_1^2 + \frac{1}{10^4}y_2^2
\end{aligned}$$

for any $x, y \in \mathbb{R}^2$ and

$$\begin{aligned}
&2x_2 f_2(x,y) + g_{21}^2(x,y) + g_{22}^2(x,y) \\
&= \frac{2\sqrt{38}}{625}x_1 x_2 - \frac{8}{5}x_2^2 + \frac{2}{10^4}x_2 y_1 + \frac{2}{10^4}x_2 y_2 \\
&\le \frac{1}{5^6}x_1^2 - \frac{399}{5000}x_2^2 + \frac{1}{10^4}y_1^2 + \frac{1}{10^4}y_2^2
\end{aligned}$$

for any $y \in \mathbb{R}^2$.

If we take

$$a_{11} = -\frac{399}{5000}, a_{12} = 2, a_{21} = \frac{1}{5^6}, a_{22} = -\frac{399}{5000},$$
$$b_{11} = \frac{1}{10^4}, b_{12} = \frac{1}{10^4}, b_{21} = \frac{1}{10^4}, b_{22} = \frac{1}{10^4},$$

and choose $\varepsilon = \frac{499}{1000} < \frac{1}{2} = \frac{\min_i a_{ii}}{d \max_i a_{ii}}, p_1 = 500, p_2 = 1$, we obtain that

$$\varepsilon a_{11} p_1 + a_{12} p_2 + b_{11} p_1 + b_{12} p_2 < 0$$
$$a_{21} p_1 + \varepsilon a_{22} p_2 + b_{21} p_1 + b_{22} p_2 < 0.$$

Since both f and g are locally Lipschitz continuous. Thus, there exists a unique global solution to equation (28), and the trivial solution is exponentially stable in mean square sense by Theorem 1. Moreover, mean square and almost sure exponential stability holds for the θ-EM method for any fixed $\theta \in (\frac{1}{2}, 1]$ by Theorem 2.

By Theorem 4, the MTEM method is also mean square and almost surely exponentially stable.

Meanwhile, we can find

$$\begin{aligned}
&2\langle x, f(x,y)\rangle + ||g(x,y)||^2 \\
=&2x_1 f_1(x,y) + 2x_2 f_2(x,y) + g_{11}^2(x,y) \\
&\quad + g_{12}^2(x,y) + g_{21}^2(x,y) + g_{22}^2(x,y) \\
=&-\frac{2}{5}x_1^2 + (\frac{2\sqrt{38}}{625} + \frac{8}{5})x_1x_2 - \frac{8}{5}x_2^2 + \frac{2}{10^4}x_1y_1 \\
&\quad + \frac{2}{10^4}x_1y_2 + \frac{2}{10^4}x_2y_1 + \frac{2}{10^4}x_2y_2.
\end{aligned} \tag{29}$$

We claim that (29) could not be written as Khasminskii-type condition (2). Indeed, it follows that

$$\begin{aligned}
&2\langle x, f(x,y)\rangle + ||g(x,y)||^2 \\
&\leq -\frac{2}{5}x_1^2 + (\frac{\sqrt{38}}{625} + \frac{4}{5})(n_1 x_1^2 + \frac{1}{n_1}x_2^2) - \frac{8}{5}x_2^2 + \frac{1}{10^4}(n_2 x_1^2 + \frac{1}{n_2}y_1^2) \\
&\quad + \frac{1}{10^4}(n_3 x_1^2 + \frac{1}{n_3}y_2^2) + \frac{1}{10^4}(n_4 x_2^2 + \frac{1}{n_4}y_1^2) + \frac{1}{10^4}(n_5 x_2^2 + \frac{1}{n_5}y_2^2) \\
&= -\frac{1}{10^4}(4000 - (16\sqrt{38} + 8000)n_1 - n_2 - n_3)x_1^2 \\
&\quad - \frac{1}{10^4}(16000 - \frac{8000 + 16\sqrt{38}}{n_1} - n_4 - n_5)x_2^2 \\
&\quad + \frac{1}{10^4}(\frac{1}{n_2} + \frac{1}{n_4})y_1^2 + \frac{1}{10^4}(\frac{1}{n_3} + \frac{1}{n_5})y_2^2.
\end{aligned}$$

for any $n_1, n_2, n_3, n_4, n_5 > 0$.

However, if (2) holds for some $C_1 > C_2 > 0$, then we must have

$$\frac{1}{10^4}\min\{4000-(16\sqrt{38}+8000)n_1-n_2-n_3,\\ 16000-\frac{8000+16\sqrt{38}}{n_1}-n_4-n_5\}\geq C_1 \tag{30}$$

for some $n_1, n_2, n_3, n_4, n_5 > 0$.
Since there is no $n_1 > 0$ such that $4000-(16\sqrt{38}+8000)n_1 > 0$ and $16000-\frac{8000+16\sqrt{38}}{n_1} > 0$, (29) can never be written in the form of Khasminskii-type conditions (2).

However, we can get mean square exponential stability of both the exact solution $x(t)$ and the θ-EM method X_k for any fixed $\theta \in (\frac{1}{2}, 1]$ by Theorem 1 and Theorem 2. And X_k is also almost surely exponentially stable.

Moreover, the MTEM method is also mean square and almost surely exponentially stable by Theorem 4.

Example 2 Consider the following 2-D stochastic differential delay equation

$$\begin{aligned} dx_1(t) &= (-\frac{2}{5}x_1(t) + \frac{4}{5}x_2(t) + \frac{1}{10^4}x_1(t-\tau(t)) + \frac{1}{10^4}x_2(t-\tau(t)))dt \\ &\quad + \frac{\sqrt{10}}{5}x_1(t)dB_1(t) \\ dx_2(t) &= (\frac{\sqrt{38}}{625}x_1(t) - x_2(t) + \frac{1}{10^4}x_1(t-\tau(t)) + \frac{1}{10^4}x_2(t-\tau(t)))dt \\ &\quad + \frac{\sqrt{10}}{5}x_2(t)dB_2(t) \end{aligned} \tag{31}$$

for $t \geq 0$, with initial value $\xi(s) \in C([-\tau, 0], \mathbb{R}^d)$, where $\tau = 0.1, \theta \in [-\tau, 0]$, $\tau(t) = 0.1|\sin(t)| \leq \tau$.

Similar to Example 1, it is easy to verify that (9) holds for

$$a_{11} = -\frac{399}{5000}, a_{12} = 2, a_{21} = \frac{1}{5^6}, a_{22} = -\frac{399}{5000},$$
$$b_{11} = \frac{1}{10^4}, b_{12} = \frac{1}{10^4}, b_{21} = \frac{1}{10^4}, b_{22} = \frac{1}{10^4},$$

and (12) holds for $\varepsilon = \frac{499}{1000}, p_1 = 500, p_2 = 1$. It is easy to see f satisfies (14).
Therefore, it follows that the θ-EM method (it is well defined since f is global Lipschitz continuous in this case) is mean square and almost surely exponentially stable for any fixed $0 \leq \theta \leq \frac{1}{2}$ by Theorem 3.

Moreover, the MTEM method is also mean square and almost surely exponentially stable by Theorem 4.

Thanks!